UNIT 4 ANGULAR MOTION

Structure

4.1 INTRODUCTION

In Units 1 to 3 you have studied some important concepts in mechanics, such as displacement, velocity. acceleration, force. linear momentum, work and energy. You have also studied two important conservation principles: Conservation of linear momentum and Conservation of energy. However, our development of the concepts of mechanics so far has been restricted in one important respect. We have not developed techniques to describe and analyse the angular motion of particles, in particular their rotational motion,

You may say that we have studied the problems of uniform circular motion and projectile motion using these concepts. But the world is full of objects that undergo rotational motion: From rotating galaxies to orbiting planets, from merry-go-rounds, bicycle wheels and flywheels to rotating ballerinas (dancers) and acrobats. In principle, we can analyse all such motions using Newton's laws by applying them to each particle of the object undergoing angular motion. But in practice it is a difficult task. especially for extended bodies, because the particles number in thousands. What we need is a simple method for treating the angular motion of an object as a whole.

In most cases. we can study the angular motion of an object in terms of the angular motion of a point on it. Therefore, in this unit we shall study the angular motion of a particle and develop related concepts, such as angular displacement, angular velocity, angular acceleration. torque and angular momentum. Using these concepts, we shall study angular motion of rigid bodies in Unit 9. In the next unit, we will turn our attention to gravitation and other forces in nature.

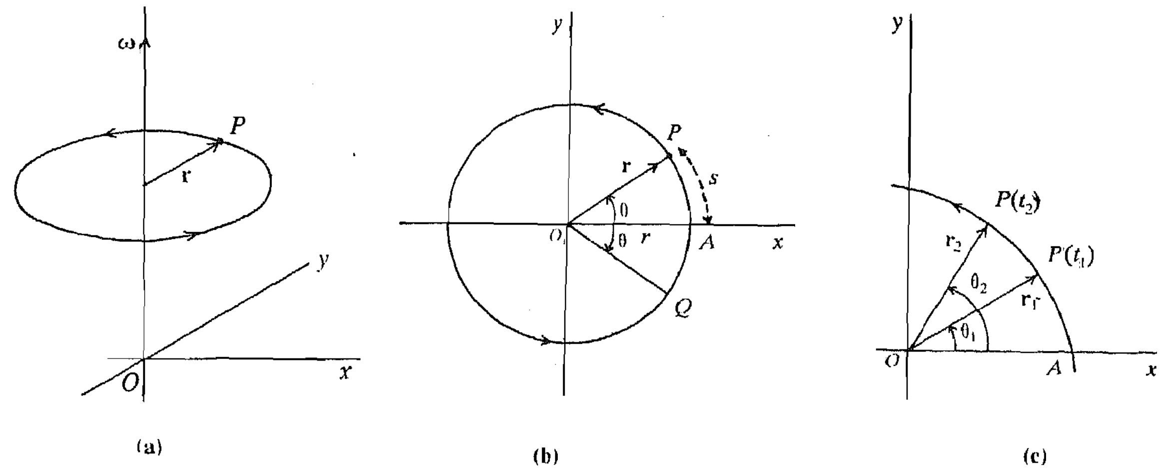
Objectives

After studying this unit you should be able to:

- compute angular displacement, angular velocity and angular acceleration of a particle undergoing angular motion
- express displacement, radial and transverse velocities, and radial and transverse acceleration using plane polar coordinates
- relate the kinematical variables of angular motion and linear motion in their vector forms
- solve problems related to the concepts of torque, rotational kinetic energy and angular momentum of a particle
- apply the law of conservation of angular momentum.

4.2 KINEMATICS OF ANGULAR MOTION

Let us begin our study of angular motion by considering a particle moving in a circle about a fixed axis passing through the centre and perpendicular to the plane of the circle. (Fig. 4.1a).



Pig. 4.1: (a) A particle P rotating anticlockwise in a circle about a fixed axis, known as the axis of rotation; (b) the angular position θ of the particle at an instant t; (c) the particle P undergoes an angular displacement $\Delta\theta$ (= $\theta_2 - \theta_1$) in time $\Delta t (= t_2 - t_1)$.

As you know from Sec. 1.4, we need only a two-dimensional frame of reference to describe this motion (Fig. 4.1b). The angle 0 is the angular position of the particle at P with respect to the reference axis, namely the x-axis. By convention, we take θ to be positive for anticlockwise rotation and negative for clockwise rotation. It is given, in radians, by the relation

$$\theta = \frac{s}{r} \qquad (4.1)$$

where s is the arc length shown in Fig, 4.1 b and r the magnitude of the position vector r of the particle. If the particle rotates more than once, then θ will take the increased value' accordingly. For example, let the particle be at P at the instant t after completing two rotations around the circle starting from *A*. Then its angular position at the instant t will be given by the angle $(2 \times 2\pi + 0) = (4\pi + 0)$. Now, let the particle rotate anticlockwise. Let its angular positions at time t, and at a later time t, be θ_1 and θ_2, respectively (see Fig. 4.1 c). The angular displacement of the particle will be 0, $-$ 6, $= \Delta\theta$ during the time interval $t_2 - t$, = *At*. Notice that we have used the term 'angular displacement'. Is this a vector quantity like linear displacement? Let us find out and discuss angular displacement in somewhat greater detail.

You are perhaps more familiar with the unit of degrees for measuring angles. The unit of radians is related to degrees by the following formula:

360° = 2 π rad;

π = 3.1415927 ...

4.2.1 Angular Displacement

If we say that angular displacement is a vector, then, firstly, alongwith a magnitude it should have a direction. Secondly, angular displacements should add like vectors. As you can see, the magnitude of the angular displacement is the angle through which the particle turns. What is the direction of angular displacemeqt?

In a sense the idea of a direction is associated with angular motion. We have both clockwise and anticlockwise rotations. Let us represent an anticlockwise rotation of say, θ rad by an arrow of a certain length pointing in a certain direction. Then a rotation of –0 rad will be an arrow of the same length, but pointing in the opposite direction. But in what direction should the first arrow point?

It obviously cannot be the direction of the particle's position vector at its final angular position. Why? See Fig. 4.1b again. For an anticlockwise rotation through an angle θ, the direction of angular displacement would be OP. But for a clockwise rotation through the same angle, its direction will be *OQ*. So, two equal and opposite rotations (clockwise and anticlockwise) of any magnitude will not in general be antiparallel. Thus, with this choice of directions, angular displacements will not be vectors.

Then how can we define the direction of angular displacement? You must have handled a screw-gauge at echool. There the rotational motion of the screw is translated into the forward motion of the screw-head which takes place along a straight line. This straight line can define the direction of the rotational motion of the screw. This straight line is essentially the axis of rotation of the screw.

So we can define the direction of angular displacement to be along the axis of rotation, But how do we represent a clockwise or an anticlockwise rotation along the axis of rotation?

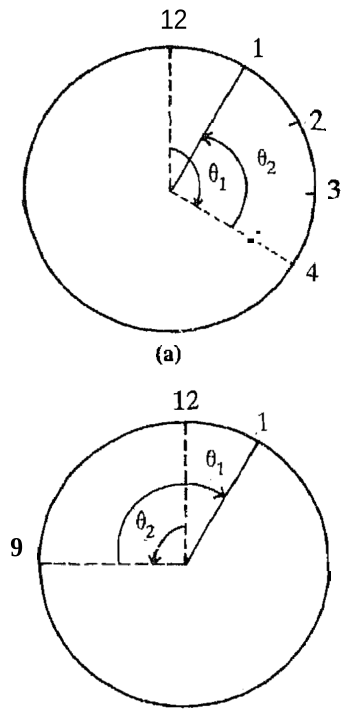


Fig. 4.2

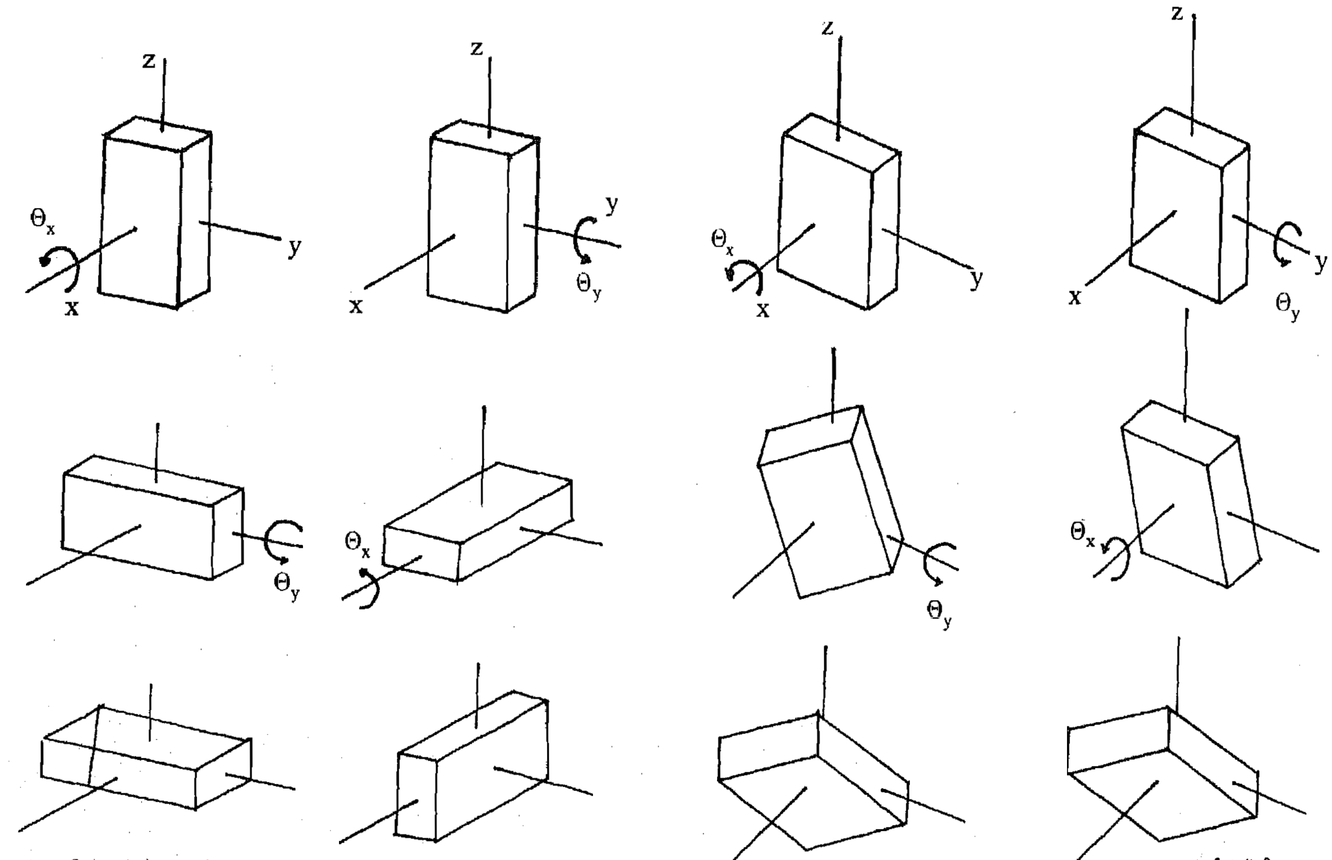
We follow the right-hand rule to make the choice. We curl the fingers of our right-hand around the axis, in the direction of rotation of the particle. The extended thumb points along the direction of the angular displacement (see Fig. 1.9b). Thus, for the particle of Fig. 4.1 (a), the direction of θ will be along the positive z-axis. In Fig. 4.1 (b), the direction of θ will be perpendicular to the page and the point up out of the page.

SAQ 1

What would be the magnitude and direction of.the angular displacement in a clockwise rotation of a hand of a clock from 5 to 9?

Having specified the direction of the angle turned by a rotating particle, let us see whether it satisfies the laws of vector addition. Let us consider the commutative law of vector addition: **A** + **B** = B + **A.** What happens in the two-dimensional case when the particle remains in the same plane.while rotating about a fixed axis? You can find the answer with the help of a clock as shown in Fig. 4.2. In Fig. 4.2a starting from 12, the clockhand is given a clockwise rotation $\theta_1 = 2\pi/3$ rad and then an anticlockwise rotation **8,** $= \pi/2$ rad to get the resultant 8, + **8,** In Fig. 4.2b the order of rotation is reversed : starting from 12, the clockhand is first rotated anticlockwise by $\pi/2$ rad and then clockwise by $2\pi/3$ rad, giving $\boldsymbol{\theta}_2$ + **8,.** The resultant is the same. Now perform a similar exercise with different magnitudes of **8,** and $\boldsymbol{\theta}_2$.

What do you conclude? Clearly, if the particle remains in the same plane and rotates about a fixed axis, the angular **displacement** is a vector quantity. Does this law hold for rotations in three dimensions? Study Fig. 4.3 and perform the rotations with the help of a book for an answer.



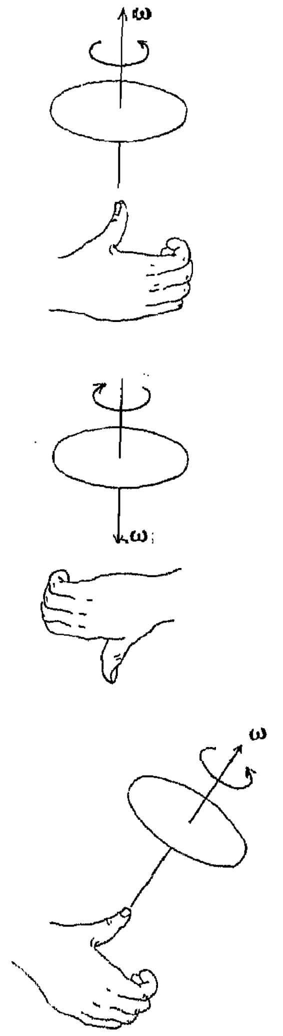
$\Theta_1 = \Theta_x\hat{i} + \Theta_y\hat{j}$ | $\Theta_2 = \Theta_y\hat{j} + \Theta_x\hat{i}$ | $\Theta_3 = \Theta_x\hat{i} + \Theta_y\hat{j}$ | $\boldsymbol{\theta}_4 = \theta_y\hat{\mathbf{j}} + \theta_x\hat{\mathbf{i}}$

(a) (b) (c) (d)

Fig. 4.3: Rotation through finite angles: (a) The book **is** rotated by an angle **of** $\pi/2$ rad **anticlockwise** around the **x-axis** ($\theta_x\hat{i}$) and **then** by $\pi/2$ anticlockwise around **the y-axis** ($\theta_y\hat{j}$). The **resultant** is $\theta_1 = \theta_x\hat{i} + \theta_y\hat{j}$; (b) the **rotations** are the **same** but in reverse order. **i.e.** $\theta_2 = \theta_y\hat{j} + \theta_x\hat{i}$. **Clearly,** $\theta_1 \neq \theta_2$. **Rotation through** infinitesimal angles: (c) the book is rotated by a small angle, **say** $\pi/36$ rad **anticlockwise** around *x* **and** *y*=**axes;** (d) the rotations are the same but in reverse order. In this case $\theta_3 = \theta_4$. In all these figures, the origin of the coordinate axes remains at the centre **of** the book, and the axes remain **parallel** to themselves.

What is the answer? Finite angular displacements in three dimensions are *not* vector quantities, but *three-dimensional infinitesimal angular displacements are vectors.*

Having defined the angular displacement and studied its vector nature, you are ready to learn about angular velocity and angular acceleration.



Pig. 4.4: The direction of the angular velocity is given by the right-hand rule.

4.2.2 Angular Velocity and Angular Acceleration

The average angular speed of a particle undergoing angular displacement *A8* in time *At* is

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} \tag{4.2a}$$

If $\Delta\boldsymbol{\theta}$ is infinitesimal, then $\bar{\boldsymbol{\omega}}$ will be a vector. It will be in the same direction as $\Delta\boldsymbol{\theta}$ and wc will call it average angular velocity. When the angular speed changes with time, we define instantaneous angular velocity as

$$\boldsymbol{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\theta}}{At} = \frac{d\boldsymbol{\theta}}{dl} \tag{4.2 b}$$

$d\boldsymbol{\theta}$ is a vector as it is an infinitesimal angular displacement. We can write $d\boldsymbol{\theta} = \frac{d\boldsymbol{\theta}}{dt}\, dt$.

Since *dt* is a scalar, $\frac{d\boldsymbol{\theta}}{dt}$ will he a vector, i.e. the instantaneous angular velocity $\boldsymbol{\omega}$ is a vector quantity. Its direction lies along the axis of rotation and its sense is given by the right-hand rule. Study Fig. 4.4 to understand the vector nature of ω better.

If the angular speed of the particle in Fig. 4.1 c is riot constant, then it has an angular acceleration. If **a**, and ω_2 are the instantaneous angular velocities of the particle at times t, and t_2, respectively, then the average angular acceleration $\bar{\alpha}$ of the particle P is defined as

$$\bar{\boldsymbol{\alpha}} = \frac{\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1}{t_2 - t_1} = \frac{\Delta\boldsymbol{\omega}}{\Delta t} \tag{4.3 a}$$

The instantaneous angular acceleration is

$$\boldsymbol{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\omega}}{A[} = \frac{d\boldsymbol{\omega}}{dl} \tag{4.3 b}$$

What is the direction of the angular acceleration? Study Fig. 4.5. If the angular velocity changes only in magnitude hut not in direction, then ω simply increases or decreases.

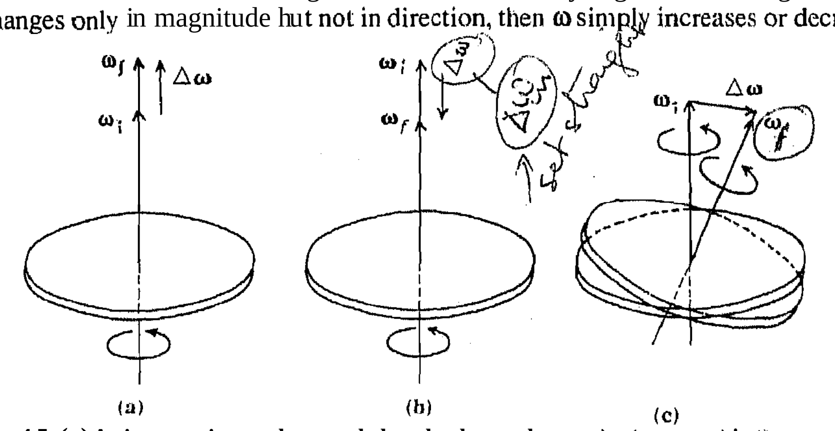


Fig. 4.5: (a) An increase in angular speed alone leads to a change Δω (= ω_f − ω_i) in the angular velocity that is parallel to a,. So a is also parallel to ω. Here a, and ω_f are the initial nnd final angular velocities, respectively; (b) a decrease in angular speed means that Δω and hence a are antiparallel to ω_i; (c) when angular velocity changes only in direction, the change Δω and hence a is perpendicular to angular velocity.

Therefore a which has a direction along $\Delta\boldsymbol{\omega}$ lies parallel or antiparallel to the axis of rotation (see Figs. 4.5 a and b). When ω changes only in direction, the angular acceleration vector is perpendicular to ω (see Fig. 4.5 c). Work out the following SAQ to prove **this** yourself.

SAQ 2

Show that a is perpendicular to **ω**, if ω is a constant. [Hint: For a to be perpendicular to **ω**, $\boldsymbol{a} \cdot \omega = 0$. Since ω is a constant, $\frac{d}{dt}(\omega^2) = \frac{d}{dt}(\boldsymbol{\omega} \cdot \boldsymbol{\omega}) = 0$.]

In most general cases, both the direction and **magnitude** of the angular **velocity** may change, in which case a is neither parallel nor **perpendicular** to **ω**.

You must have observed by now that the rotation of a particle about a fixed axis has a correspondence with the translation of a particle along a fixed direction. The kinematical variables θ, ω and α for angular motion are analogous to x, v and a for linear motion: θ corresponds to x, ω to v and α to a. You are already familiar with the relations between kinematical variables x, v, a and t for linear motion with constant acceleration. In the same manner we can derive the four equations linking θ, ω, α and t for constant angular acceleration. We are stating these relations in Table 4.1 without giving their proof.

Table 4.1: Angular and linear position, speed and acceleration

Linear Quantity or Equation	Angular Quantity or Equation
Position x	**Angular position** θ
Speed $v = \frac{dx}{dt}$	**Angular speed** $\omega = \frac{d\theta}{dt}$
Acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	**Angular acceleration** $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
Equations for Constant Acceleration	
$\bar{v} = \frac{1}{2}(v_0 + v)$	$\bar{\omega} = \frac{1}{2}(\omega_0 + \omega)$
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2ax$	$\omega^2 = \omega_0^2 + 2\alpha\theta$

Notice that you get the second set of equations merely by substituting θ for x, ω for v, α for a and the initial angular velocity ω_0 for v_0, the initial linear velocity. We have seen that a correspondence exists between linear and angular kinematical variables. Can we establish a relationship between the two sets of variables for angular motion? The answer is yes. We will find that these relations are easier to derive if we use plane polar coordinates.

4.2.3 Relating Linear and Angular Kinematical Variables

In your school mathematics courses, you may have studied plane polar coordinates r and θ of the point P (x, y), shown in Fig. 4.6a. These are related to x and y by the equations:

$$x = r\cos\theta, \; y = r\sin\theta, \tag{4.4a}$$

giving $\quad r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}.$ $\quad$ (4.4b)

You also know that

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}. \tag{4.5}$$

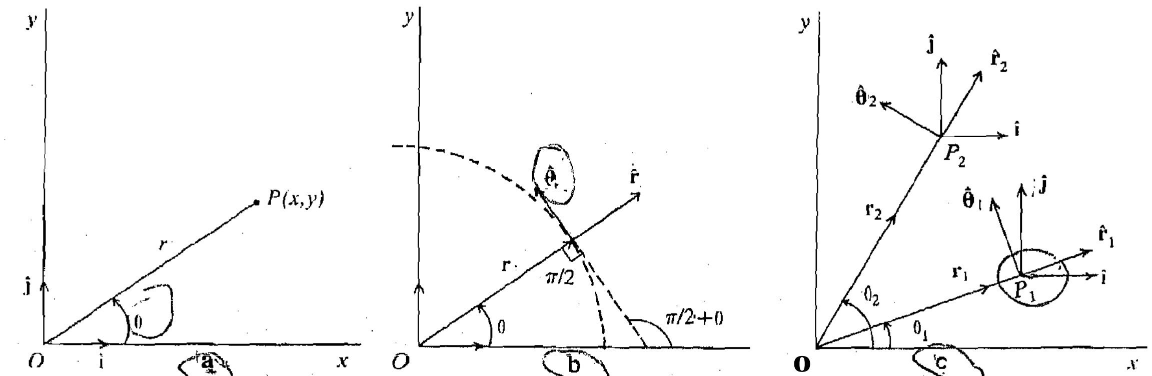


Fig. 4.6: (a) Plane-polar coordinates r and θ; (b) unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in the plane-polar coordinate system; (c) unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ have different directions at points P_1 and P_2, i.e. they vary with the position of the particle.

We now introduce two new unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, perpendicular to each other which point in the direction of increasing r and in the sense of increasing angle θ, respectively (see Fig. 4.6b). There is an important difference between the two sets of unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}})$ and $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$: $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ have fixed directions but the directions of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ vary with the position of the particle as you can see in Fig. 4.6c. Since $\hat{\mathbf{r}}$ is a unit vector along $\mathbf{r}$, we can write

$$\mathbf{r} = r\,\hat{\mathbf{r}}. \tag{4.6}$$

We can use **Eqs. 4.4, 4.5** and 4.6 to find the relationship between $\hat{\mathbf{r}}$, $\hat{\theta}$ and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$. From Eqs. **4.4, 4.5** and 4.6, we get

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{1}{r}(r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}}),$$

$$\text{or } \hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}. \qquad (4.7a)$$

So a unit vector in the direction making an angle θ with the positive x-axis is $\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$. $\hat{\boldsymbol{\theta}}$ is a unit vector making an angle $(\pi/2 + \theta)$ with positive x-axis (see Fig. 4.6b). So in order to obtain $\hat{\boldsymbol{\theta}}$ we replace θ in the expression of $\hat{\mathbf{r}}$ by $(\pi/2 + \theta)$.

$$\text{So. } \hat{\boldsymbol{\theta}} = \cos(\theta + \pi/2)\,\hat{\mathbf{i}} + \sin(\theta + \pi/2)\,\hat{\mathbf{j}},$$

$$\text{or } \hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}. \qquad (4.7b)$$

Notice that although $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ vary with position, they depend only on θ, and not on r. Before proceeding further, we suggest that you try the following SAQ to become used to the polar coordinates:

In the text, whenever we use the terms 'velocity' and 'acceleration', we mean 'linear velocity' and 'linear acceleration'.

SAQ 3

a) Show that the results $|\hat{\mathbf{r}}| = 1$, $|\hat{\boldsymbol{\theta}}| = 1$, and $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0$ are consistent with Eqs. 4.7.

b) If $\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}}$ and $\mathbf{B} = B_r\hat{\mathbf{r}} + B_\theta\hat{\boldsymbol{\theta}}$, then prove that $\mathbf{A} \cdot \mathbf{B} = A_r B_r + A_\theta B_\theta$, where r's and θ's of A and B refer to the same point in the space.

c) Show that $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}$.

Now that you are familiar with the plane polar coordinates let us first derive the expressions of velocity and acceleration for circular motion in terms of these coordinates. In **Sec. 1.4**, you have studied **these** relations for *uniform* circular motion. You know that for constant ω, $v = \omega r$ and $a_R = \frac{v^2}{r} = \omega^2 r$. Let us now consider circular motion with variable angular speed.

Velocity and acceleration for circular motion in polar coordinates

Recall that $\mathbf{v} = \frac{d\mathbf{r}}{dt}$. Now, we have from Eq. 4.6.

$$\mathbf{v} = \frac{d(\mathbf{r})}{dt} = \frac{d(r\hat{\mathbf{r}})}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = 0 + r\frac{d\hat{\mathbf{r}}}{dt} = r\frac{d\hat{\mathbf{r}}}{dt} \qquad \left(\text{Since } r \text{ is a constant, } \frac{dr}{dt} = 0\right)$$

Notice that $\frac{d\hat{\mathbf{r}}}{dt}$ is non-zero. Let us now evaluate it.

Differentiating Eq. **4.7a** with respect to time we get

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d}{dt}(\cos\theta)\hat{\mathbf{i}} + \frac{d}{dt}(\sin\theta)\hat{\mathbf{j}}, \text{ since } \hat{\mathbf{i}} \text{ and } \hat{\mathbf{j}} \text{ are constant unit vectors,}$$

$$= -\sin\theta\frac{d\theta}{dt}\hat{\mathbf{i}} + \cos\theta\frac{d\theta}{dt}\hat{\mathbf{j}}$$

$$= \dot{\theta}\,(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})$$

where we have written $\frac{d\theta}{dt}$ as $\dot{\theta}$. Using **Eq. 4.7b,** we get

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\,\hat{\boldsymbol{\theta}} \qquad (4.8)$$

Thus, for circular motion

$$\mathbf{v} = r\dot{\theta}\,\hat{\boldsymbol{\theta}}$$

$$\text{or } \mathbf{v} = r\,\omega\,\hat{\boldsymbol{\theta}}, \qquad (4.9)$$

since $\omega = \frac{d\theta}{dt}$. Thus, the velocity of a particle moving in a circle has the magnitude ω r. It is directed along $\hat{\boldsymbol{\theta}}$, which is along the tangent to the circle. You can see that Eq. 4.9 holds for uniform circular motion also.

Again differentiating **Eq.** 4.9 with respect to time, we get the acceleration for circular motion in plane polar coordinates:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dr}{dt}\dot{\theta}\hat{\boldsymbol{\theta}} + r\frac{d\dot{\theta}}{dt}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt},$$

$$= r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}, \text{ where } \ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d^2\theta}{dt^2}$$

$$\text{or } \mathbf{a} = r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

To evaluate $\frac{d\hat{\theta}}{dt}$, we differentiate Eq. 4.7b with respect to time:

$$\frac{d\hat{\theta}}{dt} = -\frac{d}{dt}(\sin\theta)\hat{\mathbf{i}} + \frac{d}{dt}(\cos\theta)\hat{\mathbf{j}}$$

$$= -\cos\theta\frac{d\theta}{dt}\hat{\mathbf{i}} - \sin\theta\frac{d\theta}{dt}\hat{\mathbf{j}}$$

$$= -\dot{\theta}(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}).$$

Using Eq. 4.7a, we get

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{\mathbf{r}}. \tag{4.10}$$

So, the acceleration of a particle moving in a circle is

$$\mathbf{a} = r\ddot{\theta}\hat{\theta} - r(\dot{\theta})^2\hat{\mathbf{r}}$$

Since $\omega = \dot{\theta}$ and $\alpha = \frac{d\omega}{dt} = \ddot{\theta}$, we get

$$\mathbf{a} = -\omega^2 r\hat{\mathbf{r}} + \alpha r\hat{\theta}, \tag{4.11 a}$$

$$= -a_R\hat{\mathbf{r}} + a_T\hat{\theta}, \tag{4.11 b}$$

$$\text{or } \mathbf{a} = \mathbf{a}_R + \mathbf{a}_T. \tag{4.11 c}$$

Thus, for circular motion a has a radial component a_R opposite to $\hat{\mathbf{r}}$ in direction, which gives the negative sign. It also has a transverse component a_T along $\hat{\theta}$. You can see that the transverse component $\mathbf{a}_T$ vanishes for uniform circular motion. You may now like to work out an SAQ to concretise these ideas.

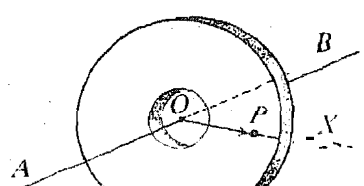


Fig. 4.7: A grindstone rotating about a fixed axis *AOB*. The particle *P* on its rim executes circular motion.

SAQ 4

A grindstone of radius 0.5 m is rotating anticlockwise at a constant angular acceleration α of 3.0 rad s^{-2} (Fig. 4.7). Start from a reference horizontal line OX at time $t = 0$, when the grindstone is at rest and find the following for a particle P situated at the rim of the

a) Its angular displacement and angular velocity 2.0 s later.

b) Its linear velocity, radial and transverse acceleration at the end of 2.0 s.

In Eqs. 4.6 to 4.11 we have expressed vectors **r**, **v** and **a** in terms of scalars θ, ω and α. What is the relation between the vectors **r**, **v**, **a** and θ, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$? Let a particle rotate in a circle about the z-axis. The vectors **r**, **v**, **a** and $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ will be as shown in Fig. 4.8. Let the angle between $\boldsymbol{\omega}$ and **r** be ϕ. Then, since ∠PCO = 90°, the radius *CP* of the circle will be $r\sin\phi$, and

$$v = \omega r \sin\phi$$

If we now sweep $\boldsymbol{\omega}$ into **r** through the smaller angle between them and use the right-hand rule, we find that the extended thumb points towards **v**. This gives the relation

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}. \tag{4.12a}$$

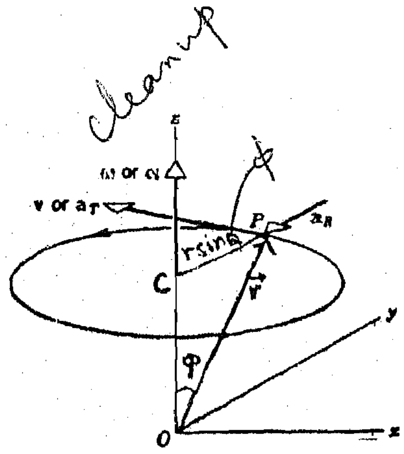


Fig. 4.8: Vectors **r**, **v**, $\mathbf{a}_T$, $\mathbf{a}_R$, $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ for a particle rotating in a *circle* about the z-axis.

Now, $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r})$

Since $\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \left(\frac{d\mathbf{A}}{dt}\right) \times \mathbf{B} + \mathbf{A} \times \left(\frac{d\mathbf{B}}{dt}\right)$, we get

$$\mathbf{a} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}.$$

We can once again prove that

$$\mathbf{a}_T = \boldsymbol{\alpha} \times \mathbf{r}, \tag{4.12b}$$

$$\mathbf{a}_R = \boldsymbol{\omega} \times \mathbf{v}, \text{ giving} \tag{4.12c}$$

$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_R \tag{4.12d}$$

Eq. **4.12b** follows from the same reasoning as we used for **v.** a, $= \alpha r \sin\phi$, and its direction is obtained from the right-hand rule ppplied to **a** and r. Now

$$a_R = \omega^2 r \sin\phi = \omega\,(\text{or}\sin\phi) = \omega v.$$

The direction of $\mathbf{a}_R$ is along **PC.** It is the same direction in which the right-hand thumb points if ω is swept into **v** through the smaller angle.

Let us now express **r, v** and **a** in terms of plane polar coordinates for any general *angular motion* of a particle about a fixed axis of rotation.

Eq. 4.6 for **r** holds good for any kind of angular motion. For velocity we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}.$$

Using Eq. 4.8, we get

$$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = \mathbf{v}_R + \mathbf{v}_T. \qquad (4.13a)$$

$$\text{where } \mathbf{v}_R = \dot{r}\,\hat{\mathbf{r}},\ \mathbf{v}_T = r\dot{\theta}\hat{\boldsymbol{\theta}} \qquad (4.13b)$$

Similarly, acceleration **a** is given as:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt},$$

$$= \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}}.$$

where we have used Eqs. 4.8 and 4.10. Thus,

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}. \qquad (4.14)$$

Eq. 4.14 means that the acceleration for general angular motion has two components. One is along $\hat{\mathbf{r}}$ and is called the radial **component.** The other is perpendicular to $\hat{\mathbf{r}}$ and is called the **transverse component.**

Eqs. 4.6 to 4.14 enable us to describe the motion of a particle undergoing angular motion either in angular variables or in linear variables. You may wonder why we need angular variables for describing angular motion, when they appear more complicated. The answer is that the angular description is more useful than the linear description when we discuss angular motion. For example, it is much more convenient to use these equations to find out the orbits of planets. You will see this in Unit 6. Similarly, for describing the motion of a rotating body we will have to consider the motion of various points on it. It is clear from Eqs. 4.6 to 4.14 that different points on the body will not have the same linear displacement, velocity or acceleration. But all points on a body rotating about a fixed axis (which does not pass through the body) have the same angular displacement, velocity or acceleration at any instant. Therefore, we can describe the motion of the whole body in a simple way if we use angular variables θ, ω and ***a.*** You will appreciate this point better when you study Unit 9. We end this section on the kinematics of angular motion with an example and an SAQ.

Example 1: Acceleration of a bead on a spoke of a wheel

A bead moves outward with a constant speed u along the spoke of a rotating wheel. It starts from the centre at time $t = 0$. The angular position of the spoke is given by $\theta = \omega t$, where ω is constant. Find the velocity and acceleration of the bead.

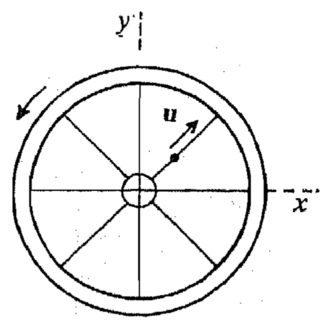


Fig. 4.9: Acceleration of a bead on a wheel's spoke

Let us choose the reference frame as shown in Fig. 4.9. Here $\dot{r} = u$ and $\dot{\theta} = \omega$. The radial position r can be obtained by integrating with respect to t the relation $\dot{r} = u$.

$$\int dr = \int u\,dt$$

or $r = ut + c$, where c = constant of integration.

At $t = 0$, $r = 0$. Thus, $c = 0$.

From Eq. 4.13

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$
$$= u\hat{\mathbf{r}} + ut\,\omega\hat{\boldsymbol{\theta}}$$
$$= \mathbf{v}_R + \mathbf{v}_T$$

We find that the magnitude of radial velocity is constant, whereas that of the transverse velocity increases linearly with time.

The **acceleration** is given by **Eq.** 4.14:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$
$$= -ut\omega^2\hat{\mathbf{r}} + 2u\omega\hat{\boldsymbol{\theta}}.$$

The magnitude of transverse acceleration is also constant.

SAQ 5

A particle moves outward along a spiral. Its trajectory is given by $r = C\theta$, where C is a constant equal to $(1/\pi)$ m rad^{-1}. θ increases in time according to $\theta = \frac{\alpha t^2}{2}$, where α is a constant.

a) Find the velocity and acceleration of the particle.

b) Show that the radial acceleration of the particle is zero when $\theta = \frac{1}{\sqrt{2}}$ rad.

[Hint: Use Eqs. 4.7, 4.13 and 4.14.].

So far we have described angular motion. We will now study the causes of angular motion.

4.3 DYNAMICS OF ANGULAR MOTION

As we have seen earlier circular motion is the simplest kind of angular motion. There are numerous examples of circular motion in nature. Many satellites are in circular orbits, the orbits of planets are nearly circular. The earth's daily rotation carries you around in circular motion. Pieces of rotating machinery, cars rounding curves etc., describe circular motion. Let us see what forces cause a particle to execute circular motion.

4.3.1 Circular Motion

We will first consider the case of uniform circular motion about which you have read in Sec. 1.4 of Unit 1. Recall that in this case, the particle moves in a circle with a constant angular speed. Thus, both r and ω are constant. The force $\mathbf{F}$ is given by Newton's second law as $\mathbf{F} = m\mathbf{a}$.

We use the expression of **a** from Eq. 4.11. In this case α is zero as ω is constant. So we get

$$\mathbf{F} = -ma_R\,\hat{\mathbf{r}} = -mr\omega^2\hat{\mathbf{r}} = -\frac{mv^2}{r}\hat{\mathbf{r}}. \qquad (4.15)$$

You can recognise the term $\frac{v^2}{r}$ as the centripetal acceleration of **Eq. 1.30c**. The force defined by **Eq. 4.15** has a magnitude mv^2/r and is directed toward the centre of the circle. The negative sign in **Eq. 4.15** appears because **F** is opposite to r in direction. This is called the centripetal force. What does **Eq. 4.15** mean? It means that for an object of mass m to be in uniform circular motion, a net force $-\frac{mv^2}{r}\hat{\mathbf{r}}$ must act on the object, Whenever we see an object in uniform circular motion, we know that a net force of this magnitude must be acting. Some physical mechanism like gravity, tension in a string, an electric or magnetic force, friction etc. must provide this force. For example, the giant planet Jupiter circles the Sun at a speed of 13 kms^{-1}. The gravitational force keeps it in its approximately circular path. Similarly, when a tiny sports cut rounds a tight curve, the centripetal force needed to keep it in a circular path is provided by the frictional force between its tyres and the roadbed, and also by the banking of the road. Protons circle around an accelerator ring because a magnetic force provides the centripetal force.

Example 2

A geostationary satellite is held in its orbit by the force of gravitation. What is its height above the surface of the earth?

You may have studied about geostationary satellites in Unit **29** of the Foundation Course FST 1. You may know that **its** time **period** of rotation is **24** h which is the same as the **period** of rotation of the Earth about its axis. Now, the centripetal force needed to keep the satellite in its **path** is provided by the force of gravitation between the Earth and the satellite. So, if

m_s and m_E are the masses of the satellite and the Earth, respectively, and r the radius of the satellite's orbit, then

$$\frac{m_s v^2}{r} = \frac{G m_s m_E}{r^2}$$

where v is orbital velocity of the satellite given as $v = \frac{2\pi}{T} r$,
and T = Time period of rotation = 24 h = 24 X 60 X 60 s

So, we get, $\frac{4\pi^2}{T^2} r = \frac{G m_E}{r^2}$ or $r^3 = \frac{G m_E T^2}{4\pi^2}$.

Putting $r = R_E + h$, where R_E = the radius of earth and h = height of the satellite above the surface of earth, we get

$$h = \left(\frac{G m_E T^2}{4\pi^2} \right)^{1/3} - R_E. \qquad (4.16)$$

Substituting the values of G, m_E and R_E and putting T = 24 x 60 X 60 s, we get

$h = 3.59 \times 10^6$ m = 35900 km.

SAQ 6

Suppose the moon were held in orbit not by gravitation of the Earth but by the tension in a massless cable. Estimate the magnitude of the tension in the cable.

What is the force for circular motion in which the angular speed of the particle changes? For example, the rotary motion of a particle on a record turntable spinning up from rest to full speed, or a ball swung in a vertical circle. In this case, we again use Eq. 4.11 for a and obtain

$$\mathbf{F} = m\mathbf{a} = \mathbf{F}_R + \mathbf{F}_T \qquad (4.17a)$$

where $\mathbf{F}_R = -mr\omega^2 \hat{\mathbf{r}} = -\frac{mv^2}{r}\hat{\mathbf{r}}$, and $\qquad (4.17b)$

$$\mathbf{F}_T = mr\alpha\,\hat{\boldsymbol{\theta}}. \qquad (4.17c)$$

Thus, for non-uniform circular motion the force has a finite transverse component in addition to the radial or centripetal component. You have studied in Sec. 1.4 of Unit 1 that the centripetal acceleration and, therefore, the centripetal force changes only the ***direction*** of velocity, and not its magnitude. What effect does the transverse force have on the particle?

Role of transverse force

The transverse force gives the particle a finite angular acceleration: the greater the force, the greater is α, and greater the rate at which angular speed increases. In other words, this force makes the particle turn faster and faster, if it continues to act. What do you think will happen to the rotating object if this force stopped acting?

If $\mathbf{F}_T$ is zero and $\mathbf{F}_R$ continues to act, the particle will continue to rotate in a circle but with zero angular acceleration, i.e. at constant angular speed. Thus, to keep a particle moving in a circle at a constant angular speed, only a centripetal force is needed. Only if you want to increase or decrease the rate at which the particle is rotating, you have to apply a transverse force in a direction perpendicular to the radius. Suppose you want to start rotating a wheel, (potter's wheel or bicycle wheel), or a grindstone or a merry-go-round, which is initially at rest (see Fig 4.10). You will have to apply a transverse force because you want to change its angular speed from zero to some positive value. You also need a centripetal force to make it move in a circle. Hence, you apply a force which is not exactly perpendicular to the radius but along the direction of the resultant **F** of the radial and transverse forces, i.e. tilted a little towards the centre of the object.

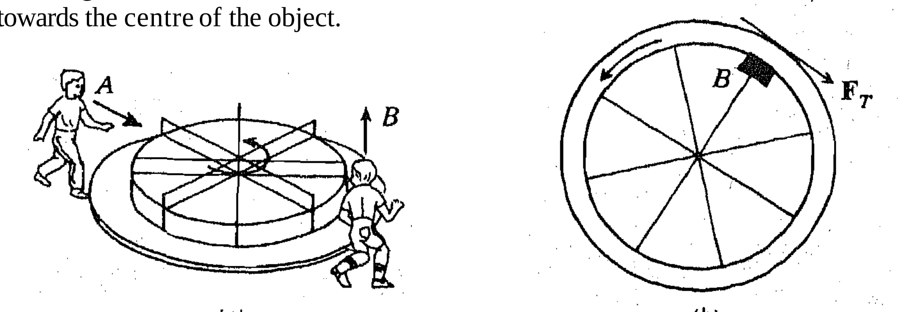


Fig. 4.10: (a) A transverse force along *B* is needed alongwith the radial force along A to set the merry-go-round moving; (b) you apply a retarding transverse force F_T while braking a bicycle wheel.

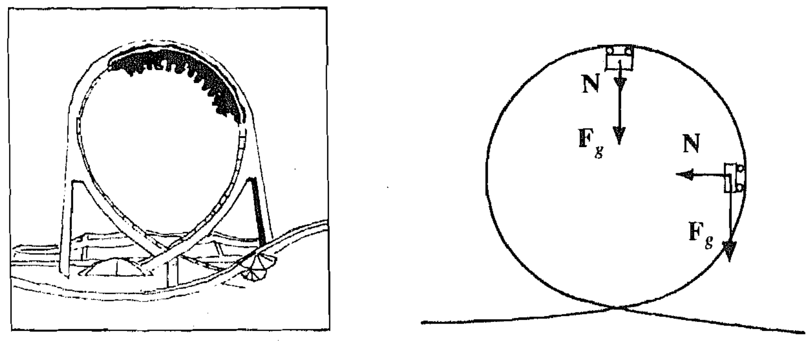
T — subscript

Activity
Try to rotate a merry-go-round, a grindstone or a bicycle wheel yourself. What is the direction in which you apply the force? Draw the direction on Fig. 4.10a.

We have seen that a transverse force is needed to increase the angular speed of a rotating object. The same force but in opposite direction would be required to reduce the angular speed of the object. This is what happens when you apply brakes while riding a bicycle. The surface of the brake B comes in contact with the rim of the wheel which rotates in an anticlockwise direction (see Fig. 4.10 b). It produces a transverse frictional force $\mathbf{F}_T$ in the opposite direction, decreasing the angular speed of the wheel.

Actually, friction is always present between a rotating wheel and the shaft or axle about which it rotates. Therefore, left to itself it will stop rotating, sooner or later due to friction. This is the same as in straight line motion where a force of friction slows down a moving object till it stops.

Example 3
A roller coaster has a Loop-the-Loop section of radius r (Fig.4.11(a)). What should the speed of a train be if it is not to leave the track even at the top of the loop?



(a) (b)

Fig. 4.11: (a) Loop-the-Loop roller coaster is a winding train track in amusement parks. Forces on the train include gravity and the normal force of reaction between the train and the track. The resultant of these forces provides the centripetal force to keep the train moving on a circular path;(b) at the top of the loop the net force on the passengers points downwards.

What are the forces acting on the train and the track? These are gravity and the normal force of reaction, between the train and track. The train will stay on the track only as long as the normal force of reaction between the train and the track remains non-zero. The forces are shown in the Fig. 4.11(b) at two points on the loop, The net force at any point is related to acceleration by Newton's second law:

$$\mathbf{F}_g + \mathbf{N} = mn.$$

Let us, for convenience, choose a coordinate system with the positive direction downward. **At** the top of the loop, the vertical component of the force equation becomes

$$mg + N = ma = \frac{mv^2}{r},$$

so that

$$v^2 = gr + \frac{Nr}{m}.$$

Now, if N is to remain non-zero at the top of the loop, then

$$(v^2 - gr) > 0,$$
$$\text{i.e. } v^2 > gr,$$
$$\text{or } v > \sqrt{gr}.$$

Therefore, for the train to be in contact with the track even at the top of the loop, its speed should always be greater than $\sqrt{gr}$. So for a typical roller coaster for which r = 6m, say, $\sqrt{gr} = \sqrt{(9.8\,\mathrm{ms^{-2}})(6\mathrm{m})} = 7.7\,\mathrm{ms^{-1}}$. The train's speed, therefore, should always be greater than 7.7 $\mathrm{ms^{-1}}$ in this case.

SAQ7
A level road has a turn of 95 m radius of curvature. What is the maximum speed with which a car can negotiate this turn (a) when the road is dry and the coefficient of static friction is 0.88 and (b) when the road is snow-covered and the coefficient of static friction is 0.21?

[Hint: The frictional force between tyres and road provides the car's acceleration.]

4.3.2 Angular Motion in General

Let us now determine the force acting on a particle executing accelerated angular motion,

From Newton's second law, using Eq. 4.14 we have

$$\mathbf{F} = m\mathbf{a} = m[\ddot{r} - r\dot{\theta}^2]\hat{\mathbf{r}} + m[r\ddot{\theta} + 2\dot{r}\dot{\theta}]\hat{\boldsymbol{\theta}} = \mathbf{F}_R + \mathbf{F}_T, \quad (4.18a)$$

where $\mathbf{F}_R$ is the radial force which acts along $\hat{\mathbf{r}}$ and has a magnitude

$$F_R = m(\ddot{r} - r\dot{\theta}^2). \quad (4.18b)$$

and $\mathbf{F}_T$ is a transverse force which acts perpendicular to $\hat{\mathbf{r}}$ and has a magnitude

$$F_T = m[r\ddot{\theta} + 2\dot{r}\dot{\theta}]. \quad (4.18c)$$

Equations 4.18 are very general. They can be used to solve any problem of motion in two dimensions, such as planetary motion. These expressions may look a little complicated to you. Don't let this put you off. All that we need to understand is this: We can use plane polar coordinates to describe any two-dimensional motion. Then, such a motion may be seen as a combination of straight line motion along the radius vector and a rotation about the origin of the frame of reference. The straight line motion is accelerated due to a radial force. The rotation, which is also an accelerated motion is the result of transverse force. For most situations Eqs. 4.18 are reduced to a simple form.

So far we have applied Newton's second law to study the angular motion of a particle. However, if the rotating object were a rigid body, then applying Newton's laws to determine the motion of every particle in it would be too cumbersome. Can we, instead, formulate an analogous law that deals directly with rotational quantities? For doing this, we need the analogues of force, linear momentum and acceleration for angular motion. We have seen that the angular acceleration is the rotational analogue of linear acceleration. What is the rotational analogue of force? The answer is torque, which we will now study.

4.3.3 Torque

Perform the following activity to understand what torque is.

Activity

Open a door by applying a force near its edge along the door's plane. Then try to open it by pushing it at the same point in a direction perpendicular to or at an angle with the door's plane. Next push it at a point near the hinges, with roughly the same force, In which case does the door open more quickly?

You can repeat this activity to open a book or to open a rusty nut with a spanner.

You would have noticed that in all cases, the job was easier if you applied the force at the point farthest away from the axis of rotation and also in a direction perpendicular to the plane of the door, or the book, or the arm of the spanner. So, how easily an object rotates depends not only on the force but also on the point and on the angle at which the force is applied, i.e. it depends on the torque. We define the torque for a single particle observed from an inertial frame of reference as follows:

If a force F acts on a particle at a point P which has a position vector $\mathbf{r}$, the torque $\boldsymbol{\tau}$ acting on the particle with reference to the origin O is defined as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \quad (4.19a)$$

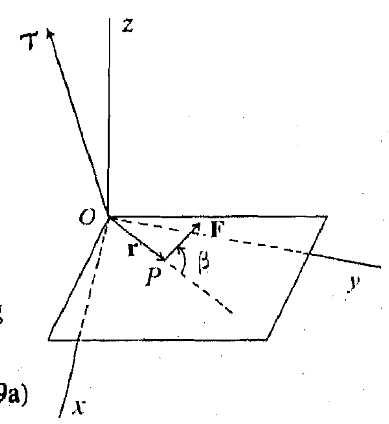


Fig.4.12: A force F is applied to a particle P, displaced r relative to the origin. F makes an angle β with r. The direction of torque is perpendicular to the plane containing r and F with the sense given by right-hand rule.

Torque is a vector quantity (Fig. 4.12). Its magnitude is given by

$$\tau = rF \sin \beta, \quad (4.19b)$$

where β is the angle between r and F. Its direction is normal to the plane formed by r and F. Thus $\boldsymbol{\tau}$ and F are always perpendicular to each other. The unit of torque is newton-metres. Now, if we substitute $\mathbf{F}$ from Eq. 4.18 (a), we get

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}_R + \mathbf{r} \times \mathbf{F}_T. \quad (4.19c)$$

Since $\mathbf{F}_R$ is parallel to r, their cross product will be zero.

$$\therefore \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}_T. \quad (4.19d)$$

It is important to realise that torque and force are entirely different quantities. The concept of torque provides a relation between the applied force and the tendency of a body to rotate.

For one thing, torque depends on the origin but force does not. You produce greater torque for the same **force**, if you **apply** the force at greater distances from the pivot point or the origin. Again, for a given force and distance r, the torque is greatest when **r** and **F** are at right angles (see Fig. 4.13a).

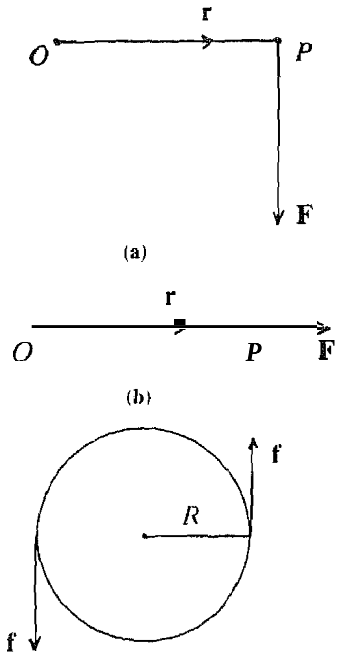


Fig 4.13:(a) The torque is greatest with F and r at right angles; (b) zero when they are collinear; (c) there can be a torque on a system with zero net force.

The torque becomes zero when **r** and **F** are along the same line (Fig. 4.13b). Thus, even if torque is zero, the external force need not be zero. The torque is also zero if the force acts on the point or along the axis, about which the particle is rotating. This is because in such a case, the vector **r** will be zero. Torque is obviously zero if the external force itself is zero. There can also be a torque on a system with zero net force (Fig. 4.13c). In general there will be both torque **and** force.

Let us now find out the torque acting on a particle in circular motion in the xy-plane. Using Eq. 4.17 c, and Eq. 4.19 d we get

$$\begin{aligned} \tau &= \mathbf{r} \times mr\alpha\hat{\theta} \\ &= mr^2\alpha(\hat{\mathbf{r}} \times \hat{\theta}), \text{ since } \mathbf{r} = r\hat{\mathbf{r}} \\ &= mr^2\alpha\hat{\mathbf{k}}, \text{ since } \hat{\mathbf{r}} \times \hat{\theta} = \hat{\mathbf{k}}, \text{ from SAQ 3(c)}. \\ &= mr^2\boldsymbol{\alpha}, \end{aligned} \tag{4.20}$$

since $\alpha\hat{\mathbf{k}}$ is simply the angular acceleration vector $\boldsymbol{\alpha}$.

Let us compare Eq. 4.20 with Newton's second law $\mathbf{F} = m\mathbf{a}$. The torque is the product of the angular acceleration $\boldsymbol{\alpha}$ and a quantity mr^2. On comparison we can say that this quantity mr^2 is the rotational analogue of the mass. We call the quantity mr^2 the **rotational inertia** or **moment of inertia** and represent it by the symbol I. Rotational inertia has the units kg m^2 and accounts both for mass of the particle and for the location of the particle relative to the axis of rotation. You know that the inertial mass is a measure of the body's resistance to change in its state of motion. In the same way, rotational inertia is a measure of the body's resistance to change in its rotational motion. Note that I would change if we change the axis of rotation. In contrast m is a constant. Substituting I for mr^2 in Eq. 4.20, we can write for circular motion of a particle of mass m about a fixed axis of rotation

$$\tau = I\,\alpha, \tag{4.21a}$$

$$\text{where } I = mr^2. \tag{4.21b}$$

This equation is similar to Newton's second law. We can deduce the same kinds of things from it as we did from equation $\mathbf{F} = m\mathbf{a}$. For instance, for constant I, the angular acceleration is directly proportional to the applied torque. In the absence of torque, an object continues to move at a constant angular speed. And, the same torque will produce greater angular acceleration for an object of smaller moment of inertia. You can now apply the Eqs. 4.21a and 4.21b to solve a problem in which the torque acts to change the particle's angular velocity.

SAQ 8

You may have studied in Block 3 of Foundation Course FST 1 that a neutron star is an extremely dense, rapidly spinning object that results from the collapse of a star at the end of its life. A neutron star of mass 15×10^{30} kg has a rotational inertia of 45×10^{36} kg m^2 about an axis of rotation passing through its centre. The neutron star's rotation rate slowly decreases as a result of torque associated with magnetic forces. If the rate of change in its angular speed is 5×10^{-5} rad s^{-2}, what is the magnitude of magnetic torque?

Another question concerning angular motion is whether we can express the kinetic energy of a rotating particle in terms of the angular variables? Yes, we can. Let us see how to do it.

4.3.4 Kinetic Energy of Rotation

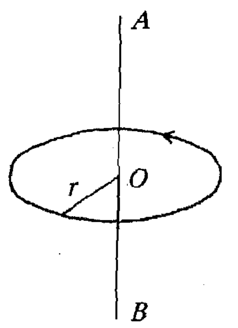


Fig. 4.14

Let us consider a particle of mass m moving in a circle of radius r about a fixed-axis of rotation AOB (see Fig. 4.14). Let its angular speed about the axis be ω. Its kinetic energy is

$$\begin{aligned} \text{K. E.} &= \frac{1}{2}mv^2 = \frac{1}{2}m(r\omega)^2, \\ &= \frac{1}{2}mr^2\omega^2 \end{aligned}$$

Thus, using Eq. 4.21b, we get

$$K_{Rot} = \frac{1}{2}I\omega^2. \tag{4.22}$$

This is also termed as the **kinetic energy of rotation** of the body.

So far, we have studied some concepts of angular motion. We have seen that an analogy exists between the kinematics and dynamics of linear and angular motion. This analogy would be complete if we could define a physical quantity corresponding to linear momentum. Indeed, there is such a quantity called angular momentum. We will discuss angular momentum especially so as to arrive at another very important conservation law.

4.4 ANGULAR MOMENTUM

We know that the torque on a particle due to a force F is given as2 = r X F. Since F = $\frac{d\mathbf{p}}{dr}$ from Newton's second law, we get

$$\boldsymbol{\tau} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}.$$

$$\text{Now, } \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{d\mathbf{r}}{dt} \times (m\mathbf{v}) + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \quad (\because \mathbf{p} = m\mathbf{v})$$

$$= 0 + \mathbf{r} \times \frac{d\mathbf{p}}{dr} \quad (\because \mathbf{v} \times niv = 0).$$

So, we can write,

$$\boldsymbol{\tau} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}).$$

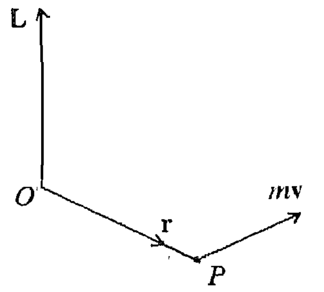


Fig. 4.15

We define the angular momentum L of the particle with respect to the origin O to be

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{4.23a}$$

Thus, angular momentum is a vector with magnitude

$$L = rp \sin \gamma, \tag{4.23b}$$

where γ is the angle between **r** and p. The direction of I, is perpendicular to the plane formed by **r** and **p.** It is determined by the right-hand rule (see Fig. 4.15). Although L has been drawn through the origin, this location has no special significance. Only the direction and magnitude of **L** are important. The unit of angular momentum is kg $m^2 s^{-1}$. Thus, the expression for torque becomes

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}. \tag{4.24}$$

You can see that this relation is analogous to Newton's second law. We can also relate angular momentum to angular velocity. Let a particle of mass m move anticlockwise in the xy-**plane** about a fixed axis of rotation perpendicular to the plane with a linear momentum **p.** Then you know that its angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} = m\mathbf{r} \times \mathbf{v}.$$

Using Eq. 4.6 for r and Eq. 4.13 for v we get

$$\mathbf{L} = m\mathbf{r} \times (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}),$$

$$= 0 + mr^2\dot{\theta}(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}), \text{ since } \hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0. \tag{4.25a}$$

$$\therefore \mathbf{L} = mr^2\, \dot{\theta}\hat{\mathbf{k}}$$

Now mr^2 is the moment of inertia **I** of the particle and $\dot{\theta}\hat{\mathbf{k}}$ is the angular velocity vector $\boldsymbol{\omega}$.

Therefore, we can write

$$\mathbf{L} = I\,\boldsymbol{\omega}. \tag{4.25b}$$

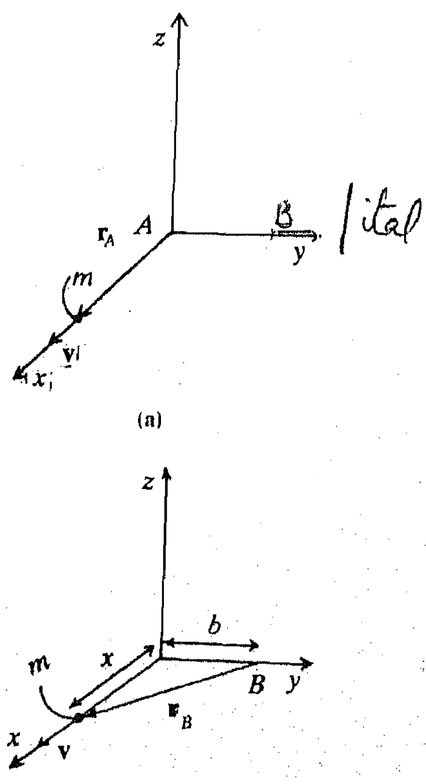
Nbtice that this equation is analogous to p = $m\mathbf{v}$.

Let us now work out an example on angular momentum.

Example 4: Angular momentum of a particle in uniform motion

A block of mass m and negligible dimensions moves at a constant speed v in a straight line (see Fig. 4.16). What is its angular momentum $\mathbf{L}_A$ about the origin A and its angular momentum $\mathbf{L}_B$ about the origin B?

Let the particle move along the x-axis, i.e., $\mathbf{v} = v\,\hat{\mathbf{i}}$. As shown in Fig. 4.16(a), the position vector of the particle with respect to A is



(a)

(b)

Fig. 4.16

$$\mathbf{r}_A = x\hat{\mathbf{i}}$$

Since $\mathbf{r}_A$ is parallel to $\mathbf{v}$, their cross product is zero and

$$\mathbf{L}_A = \mathbf{r}_A \times m\mathbf{v} = m\mathbf{r}_A \times \mathbf{v} = 0.$$

The particle's angular momentum with respect to B is

$$\mathbf{L}_B = m\mathbf{r}_B \times \mathbf{v}$$

We can write

$$\mathbf{r}_B = x\hat{\mathbf{i}} - b\hat{\mathbf{j}}$$

where x is the component of $\mathbf{r}_B$ parallel to v and b its component perpendicular to **v.**

Since $\hat{\mathbf{i}} \times \mathbf{v} = 0$, only $b\hat{\mathbf{j}}$ contributes to $\mathbf{L}_B$. Thus,

$$\mathbf{L}_B = m(x\hat{\mathbf{i}} - b\hat{\mathbf{j}}) \times v\hat{\mathbf{i}} = 0 - mbv\hat{\mathbf{j}} \times \hat{\mathbf{i}}$$

$$\mathbf{L}_B = mbv\hat{\mathbf{k}}.$$

Thus, $\mathbf{L}_B$ lies in the positive z-direction and has a magnitude mbv. This example shows how L depends on the choice of the origin. Further, for the particle moving in a straight line, b is constant. Therefore, the angular momentum of a particle moving at a constant speed in a straight line remains constant.

So, the torque acting on such a particle is zero.

Another idea brought out by the above example is this: Do not think that the quantities ω, L, α and τ can be defined, or have meaning only for angular motion. Any moving object can possess an angular velocity, angular acceleration, angular momentum and torque about an origin. What is more, the same object can have different values for these quantities about different origins.

SAQ 9

A particle of mass m falls from rest in the earth's gravitational field according to Galileo's law $z = z_0 - \frac{1}{2} gt^2$. Its horizontal coordinates are $x = x_0$, $y = 0$.

a) Determine the position vector **r** and velocity v of the particle at time t.

b) Find the angular momentum **L** as a function of time about the origin.

c) Determine the torque acting on the particle about the origin, (Hint: $\tau = \frac{d\mathbf{L}}{dt}$)

4.4.1 Conservation of Angular Momentum and its Applications

What happens when the net external torque on the particle is zero? Eq. **4.24** becomes

$$\tau = \frac{d\mathbf{L}}{dt} = 0$$

i.e. L = constant.

Thus, we get the principle of conservation of angular momentum. ***The angular momentum of a particle remains constant both in magnitude and direction if no net external torque acts on it.***

Constant angular momentum implies that the particle's motion is confined to a fixed plane normal to L. This is because by definition $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and L is normal to the plane containing r and p. Since L is constant in direction, r and v will lie in a fixed plane normal to the constant vector L. So we need to use only a two-dimensional coordinate system to study the particle's motion. The principle of conservation of angular momentum applies to systems ranging from subatomic particles to huge rotating galaxies. Let us study some applications of the law of conservation of angular momentum to understand it better.

Pointing a Satellite

Angular momentum conservation is used to steer a satellite, i.e. to point it in any desired direction. For this purpose wheels are **fixed** inside the satellite, Each wheel has a motor and brakes to start and stop its rotation. When a wheel starts rotating, the satellite rotates in the opposite direction to conserve the angular momentum. After the satellite has rotated through the desired angle, the wheel is stopped and the satellite also stops rotating. Three wheels are normally used so that the satellite can be pointed in any direction. The motors and brakes run on electricity generated through solar energy, so there is no fuel to run out.

It is also because of the conservation of angular momentum that a satellite's axis of rotation remains fixed in space. Satellites are usually rotationally isolated bodies. So the net torque acting on them is zero. Thus, the direction of **L** and hence the direction of the axis of rotation remains fixed. Therefore, spinning the satellite gives it a stability in orbit (Fig. 4.17).

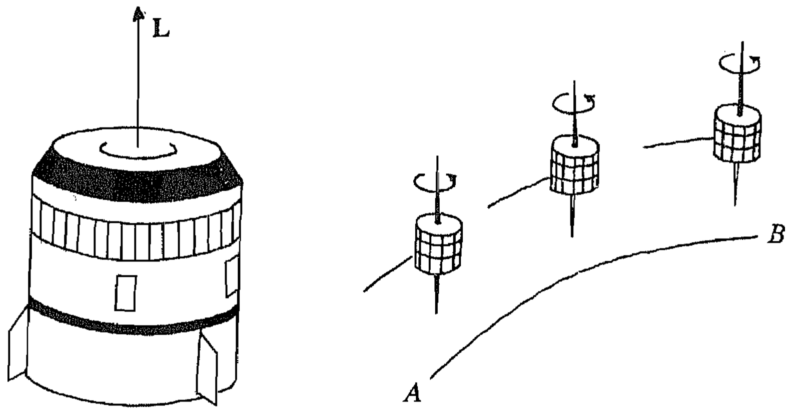


Fig. 4.17: (a) For a rotationally isolated satellite, L remains constant in magnitude and its direction remains fixed in space. Thus, the axis of rotation remains fixed as it is along L; (b) the fact that the axis of rotation remains fixed in space for constant L is used for stabilisation of a satellite by spinning it. *AB* shows a section of the earth's surface.

Angular acceleration accompanying contraction of a string

An object of mass m is attached to a string and is rotated in a horizontal plane (the plane of the dashed line in Fig. 4.18).

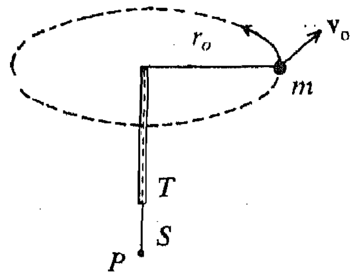


Fig. 4.18: Mass m describes circular motion of radius r_0 and velocity v_0. It is connected to a string S which passes through a tube T. The radius of the circle can be shortened by pulling on the string at P.

The object rotates with velocity $\mathbf{v}_0$ when the radius of the circle is r_0. It is seen that as the string is shortened by pulling it in, the object speeds up even as it rotates. Why does the object speed up?

The force on the object due to the string is radial. Here we are neglecting the force of gravity. Thus, the net external torque on the object is zero and its angular momentum is conserved. Therefore, as the string is shortened, the angular momentum should remain constant. The magnitude of the initial angular momentum of the object when the radius of the circle is r_0 is $|m\mathbf{r}_0 \times \mathbf{v}_0| = mr_0v_0 \sin 90° = mr_0v_0$.
The magnitude of the object's angular momentum when the radius of the circle is shortened to r is $|m\mathbf{r} \times \mathbf{v}| = mrv \sin 90° = mrv$.

Since angular momentum is constant, $mr_0 v_0 = mrv$.

This gives

$$v = \frac{v_0 r_0}{r}.$$

As r is smaller than r_0, v will be greater than v_0, that is the object will speed up.

Let us now summarise the unit.

4.5 SUMMARY

- Infinitesimal angular displacements are vectors. The angular velocity and angular acceleration vectors are defined as

$$\boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}, \quad \boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}$$

  The directions of angular displacement and angular velocity vectors are taken along the axis of rotation, and their sense is determined by the right-hand rule.

- Plane polar coordinates can be used to describe angular motion in two dimensions and to express the relationship between kinematical variables of linear and angular motion.

- For *uniform circular motion*, r and ω *are constant*.

  $\mathbf{r} = r\hat{\mathbf{r}},\ \mathbf{v} = r\omega\hat{\boldsymbol{\theta}},\ \mathbf{a}_R = -\frac{v^2}{r}\hat{\mathbf{r}}.$

  For *circular motion r is constant, ω varies giving a finite α and*

  $\mathbf{r} = r\hat{\mathbf{r}},\ \mathbf{v} = r\omega\hat{\boldsymbol{\theta}},\ \mathbf{a} = -\frac{v^2}{r}\hat{\mathbf{r}} + \alpha r\hat{\boldsymbol{\theta}}.$

- For *general angular motion, r is a variable*

  $\mathbf{r} = r\hat{\mathbf{r}},\ \mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$

  $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} = \mathbf{a}_R + \mathbf{a}_T$

- The vector forms of these relationships are

  $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r},\ \mathbf{a}_R = \boldsymbol{\omega} \times \mathbf{v},\ \mathbf{a}_T = \boldsymbol{\alpha} \times \mathbf{r}$

- Torque and moment of inertia are the analogues of force and inertial mass for angular motion. The torque acting on a particle displaced by r under the influence of force F around the origin is given by

  $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$

- There exists a relationship analogous to Newton's second law between torque and angular momentum:

  $\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$, where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

- For a particle of mass m moving in a circle of radius r around a fixed axis of rotation $\tau = I\alpha$, where $I = mr^2$ is its moment of inertia.

- The kinetic energy of a particle of mass m rotating with an angular speed ω is

  $K_{rot} = \frac{1}{2} I\omega^2$.

- If the net external torque acting on a system is zero, the angular momentum of the system is constant both in magnitude and direction. This is the principle of conservation of angular momentum and it has many applications.

4.6 TERMINAL QUESTIONS

1. Take a rectangular coordinate system. A particle moves parallel to x-axis with a constant speed v. Show that the magnitude of its angular velocity varies inversely as the square of its distance from the origin. Also obtain an expression for the magnitude of its angular acceleration.

2. A particle of mass 5g moves in a plane with constant radial speed $\dot{r} = 4\ \mathrm{m\ s^{-1}}$. The angular velocity is constant and has magnitude $\dot{\theta} = 2\ \mathrm{rad\ s^{-1}}$. When the particle is 3 m from the origin, find the (a) velocity, (b) acceleration and (c) kinetic energy of the particle.

3. A panicle of mass m moves along a space curve defined by $\mathbf{r} = 6t^4\hat{\mathbf{i}} - 3t^2\hat{\mathbf{j}} + (4t^3 - 5)\hat{\mathbf{k}}$. Find its (a) angular momentum, (b) torque and (c) kinetic energy of rotation about the origin.

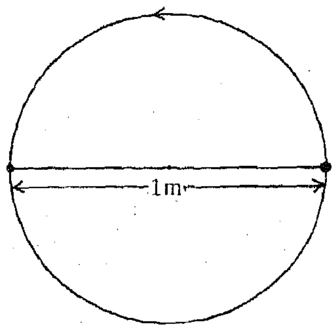


Fig. 4.19

4. Two objects of mass 20g and 30g are connected by a light rod of length 1 m and move in a horizontal circle as shown in Fig, 4.19. The speed of each is 2 m $\mathrm{s^{-1}}$. (a) What is the total angular momentum of the objects about the centre? (b) If the rod contracts uniformly to half of its original length, will the speed of the objects change? If so, by how much?

4.7 ANSWERS

SAQs

1. The magnitude of the rotation will be $2\pi/3$ rad. Its direction will be perpendicular to the face of the clock pointing away from you if you are holding it face up.

2. Since ω is constant, $\frac{d\omega}{dt} = 0$, or $\frac{d(\omega^2)}{dt} = \frac{d}{dt}(\boldsymbol{\omega} \cdot \boldsymbol{\omega}) = 0.$

Since $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \frac{dA}{dt}\cdot \mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{dt}$,

$$\frac{d}{dt}(\boldsymbol{\omega}\cdot\boldsymbol{\omega}) = \frac{d\boldsymbol{\omega}}{dt}\cdot\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\frac{d\boldsymbol{\omega}}{dt} = 2\frac{d\boldsymbol{\omega}}{dt}\cdot\boldsymbol{\omega}, \ (\because \mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A})$$

$$\therefore \frac{d}{dt}(\boldsymbol{\omega}\cdot\boldsymbol{\omega}) = 2\boldsymbol{\alpha}\cdot\boldsymbol{\omega}, \text{ or } 2\boldsymbol{\alpha}\cdot\boldsymbol{\omega} = 0$$

This implies that $\boldsymbol{\alpha}$ is perpendicular to $\boldsymbol{\omega}$.

3. a) i) $|\hat{\mathbf{r}}| = \sqrt{\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}} = \sqrt{(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})\cdot(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})}$

$$= \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

ii) $|\hat{\boldsymbol{\theta}}| = \sqrt{\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}}} = \sqrt{(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})\cdot(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$

iii) $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}} = (\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}).(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}) = -\cos\theta\ \sin\theta + \sin\theta\ \cos\theta = 0$

b) $\mathbf{A}\cdot\mathbf{B} = (A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}})\cdot(B_r\hat{\mathbf{r}} + B_\theta\hat{\boldsymbol{\theta}})$

$$= A_rB_r(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}) + A_rB_\theta(\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}}) + A_rB_r(\hat{\boldsymbol{\theta}}.\hat{\mathbf{r}}) + A_\theta B_\theta(\hat{\boldsymbol{\theta}}.\hat{\boldsymbol{\theta}})$$

$$= A_rB_r(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}) + A_\theta B_\theta(\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}}) \qquad (\because \hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{r}} = 0)$$

From a.i) and ii) $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = 1$ and $\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = 1$

Therefore, $\mathbf{A}\cdot\mathbf{B} = A_rB_r + A_\theta B_\theta$

c) $\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = (\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})\times(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})$

$$= \cos^2\theta(\hat{\mathbf{i}}\times\hat{\mathbf{j}}) - \sin^2\theta(\hat{\mathbf{j}}\times\hat{\mathbf{i}}) \qquad (\because \hat{\mathbf{i}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}\times\hat{\mathbf{j}} = 0)$$

$$= (\cos^2\theta + \sin^2\theta)\hat{\mathbf{k}} = \hat{\mathbf{k}} \qquad (\because \hat{\mathbf{i}}\times\hat{\mathbf{j}} = -\hat{\mathbf{j}}\times\hat{\mathbf{i}} = \hat{\mathbf{k}})$$

4. We shall use the equations for constant angular acceleration given in Table 4.1 along with Eqs. 4.6 to 4.11. Hence $r = 0.5$m, $\alpha = 3.0$ rad s^{-2}.

a) The magnitude of the angular displacement is given by

$$\theta = \omega_o t + \tfrac{1}{2}\alpha t^2 = \tfrac{1}{2}\alpha t^2, \ \because \omega_o = 0 \text{ in this case}$$

or $\theta = \frac{1}{2}(3\text{ rad s}^{-2})\times(2^2\text{s}^2) = 6$ rad

The direction of θ will point along the axis of rotation from O to A (Fig. 4.7). The angular speed

$$\omega = \omega_o + \alpha t = \alpha t = 3\text{ rad s}^{-2}\times 2\text{s}$$

or $\omega = 6$ rad s^{-1}

and the direction of angular velocity is along OA.

b) The linear velocity v is given by

$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = r\omega\hat{\boldsymbol{\theta}}$ since r is constant,

or v = 0.5m x 6 rad s^{-1} $\hat{\boldsymbol{\theta}}$ since $\omega = 6$ rad s^{-1} at $t = 2$s.

So the linear velocity of the particle has a magnitude 3m s^{-1} and is directed along the tangent at that point,

Radial acceleration $\mathbf{a}_R = -\omega^2 r\hat{\mathbf{r}} = -(6\text{ rad s}^{-1})^2\times 0.5\text{ m}\hat{\mathbf{r}}$

$$5 - 18\text{ m s}^{-2}\hat{\mathbf{r}}.$$

Transverse acceleration $\mathbf{a}_T = \alpha r\hat{\boldsymbol{\theta}}$

$$= (3.0\text{ rad s}^{-2})\times(0.5\text{m})\hat{\boldsymbol{\theta}} = 1.5\text{ m s}^{-2}\hat{\boldsymbol{\theta}}$$

You must note that radian is the unit of angle which is dimensionless and hence its multiplication with any other unit leaves it unchanged.

5 The trajectory of the particle is given by

$$r = C\theta = \left(\frac{1}{\pi}\text{ m rad}^{-1}\right)\left(\frac{\alpha}{2}t^2\text{ rad}\right) = \frac{\alpha t^2}{2\pi}\text{m}$$

a) The velocity $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$ from Eq. 4.13 a.

Since $\dot{r} = \frac{dr}{dt} = \frac{\alpha t}{\pi}$ m s^{-1} and $\dot{\theta} = \frac{d\theta}{dt} = \alpha t$ rad s^{-1}

$$\mathbf{v} = \left(\frac{\alpha t}{\pi}\hat{\mathbf{r}} + \frac{\alpha t^2}{2\pi}\cdot \alpha t\,\hat{\boldsymbol{\theta}}\right) \text{m s}^{-1}$$

$$= \frac{\alpha t}{\pi}\left(\hat{\mathbf{r}} + \frac{\alpha t^2}{2}\hat{\boldsymbol{\theta}}\right) \text{m s}^{-1}$$

Acceleration $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$

Since $\ddot{r} = \frac{d\dot{r}}{dt} = \frac{\alpha}{\pi}\text{ms}^{-2}$, $\ddot{\theta} = \alpha$ rad s^{-2}

$$\mathbf{a} = \left(\frac{\alpha}{\pi} - \frac{\alpha t^2}{2\pi}\cdot\alpha^2 t^2\right)\hat{\mathbf{r}} + \left(\frac{\alpha t^2}{2\pi}\cdot\alpha + \frac{2\alpha t}{\pi}\cdot\alpha t\right)\hat{\boldsymbol{\theta}}$$

$$= \left(\frac{\alpha}{\pi} - \frac{\alpha^3}{2\pi}t^4\right)\hat{\mathbf{r}} + \frac{5}{2\pi}\alpha^2 t^2\hat{\boldsymbol{\theta}}.$$

b) $a_r = 0$ means that

$$\frac{\alpha}{\pi} - \frac{\alpha^3}{2\pi}t^4 = 0, \quad \text{or } \alpha^3 t^4 = 2\alpha.$$

Since $\alpha \neq 0$, we get $\alpha^2 t^4 = 2$ or $\left(\frac{\alpha t^2}{2}\right)^2 \cdot 2 = 1$

i.e. $\theta^2 = \frac{1}{2}\text{rad}^2$ giving $\theta = \frac{1}{\sqrt{2}}$rad.

6. The tension in the massless cable holding the moon will provide the centripetal force $\frac{mv^2}{r}$. Now, if the moon were held by the force of gravitation between the earth and the moon then,

$$\frac{mv^2}{r} = \frac{Gmm_E}{r^2}$$

where m and m_E are the masses of the moon and the earth, respectively, and r is the mean distance between the moon and the earth.

So, the tension T in the cable $= \frac{mv^2}{r} = \frac{Gmm_E}{r^2}$

or $T = \frac{(6.673 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}) \times (7.35 \times 10^{22}\text{kg}) \times (5.97 \times 10^{24}\text{kg})}{(3.85 \times 10^8\text{m})^2}$

$= 1.92 \times 10^{20}$ N.

Compare this with the tensions in cables required to lift cars or trucks which are of the order of 20,000 N.

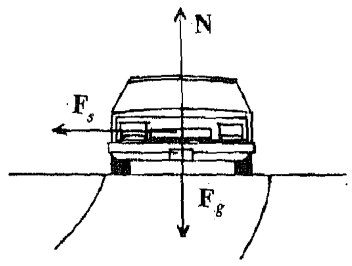


Fig. 4.20: N is the normal reaction which balances the weight F_g (=mg), F_s, the force of friction, provides the necessary centripetal force.

7. The centripetal force is provided by the force of friction between the car's tyres and the road (Fig. 4.20).

The magnitude of the force of friction $F_s = \mu_s N = \mu_s mg$, where m = the mass of the car.

So $\mu_s mg = \frac{mv^2}{r}$, where v = the maximum possible speed of the car.

$\therefore v = \sqrt{\mu_s rg}$.

a) For the dry road. $v = \sqrt{(0.88)(95\text{m})(9.8\text{ms}^{-2})} = 29\text{ms}^{-1}$,

b) For the snow covered road $v = \sqrt{(0.21)(95\text{m})(9.8\text{ms}^{-2})} = 14\text{ms}^{-1}$,

If this speed is exceeded, the car must move in a path of greater radius which means it will go off the road.

8. The torque of an object is related to its angular acceleration by $\tau = I\alpha$. Here

$I = 45 \times 10^{36}$ kg m^2 and $\alpha = 5 \times 10^{-5}$ rad s^{-2}.

Therefore, the magnitude of the magnetic torque $= (45 \times 10^{36}\text{ kg m}^2) \times (5 \times 10^{-5}\text{ rad s}^{-2})$ $= 2.2 \times 10^{33}$ newton-metres.

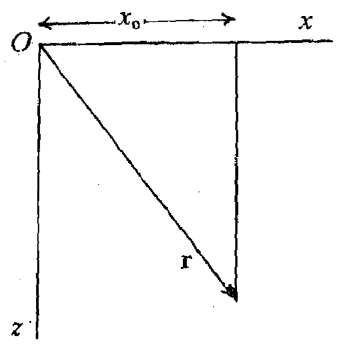


Fig. 4.21

9. a) Refer to Fig, 4.21. The position vector **r** of the particle with respect to origin at time t is

$$\mathbf{r} = x_0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + \left(z_0 - \frac{1}{2}gt^2\right)\hat{\mathbf{k}}.$$

$$= x_0\hat{\mathbf{i}} + \left(z_0 - \frac{1}{2}gt^2\right)\hat{\mathbf{k}}.$$

Its velocity at time t is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -gt\hat{\mathbf{k}}.$$

b) The angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} = m(\mathbf{r} \times \mathbf{v})$$

or $$\mathbf{L} = m\left[x_0\hat{\mathbf{i}} + \left(z_0 - \frac{1}{2}gt^2\right)\hat{\mathbf{k}}\right] \times \left[-gt\,\hat{\mathbf{k}}\right]$$

$$= m\left[x_0gt\hat{\mathbf{j}} + \mathbf{0}\right] \quad \left[\because \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}, \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}\right]$$

$$= mx_0gt\hat{\mathbf{j}}.$$

c) The torque acting on the particle about the origin is

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = mx_0g\hat{\mathbf{j}}.$$

Terminal Questions

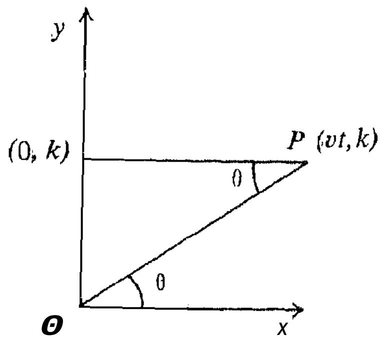


Fig. 4.22

1. Refer to Fig. 4.22. Let the particle be at P at the time t. Its distance from the y-axis is then equal to vt. So $\theta = \tan^{-1}(k/vt)$. The magnitude of its angular velocity is given by

$$\omega = \frac{d\theta}{dt} = \left[\frac{1}{1 + k^2/v^2t^2}\right]\left(-\frac{k}{vt^2}\right) = \frac{-kv}{k^2 + v^2t^2},$$

or, $\omega = \dfrac{\text{a constant}}{OP^2}$. So, ω is inversely proportional to OP^2.

The magnitude of angular acceleration is given by

$$\alpha = \frac{d\omega}{dt} = \frac{2kv^3t}{(k^2 + v^2t^2)^2}.$$

2, a) The linear velocity of the particle is

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

Here $\dot{r} = 4\ \text{m s}^{-1}$, $\dot{\theta} = 2\ \text{rad s}^{-1}$ and $r = 3\text{m}$. So,

$$\mathbf{v} = (4\ \text{m s}^{-1})\,\hat{\mathbf{r}} + (3\ \text{m} \times 2\ \text{rad s}^{-1})\hat{\boldsymbol{\theta}}$$

$$= (4\hat{\mathbf{r}} + 6\hat{\boldsymbol{\theta}})\ \text{m s}^{-1} = \mathbf{v}_R + \mathbf{v}_T$$

The magnitude of the velocity $v = \sqrt{v_R^2 + v_T^2}$

$$= \sqrt{4^2 + 6^2}\ \text{m s}^{-1} = 2\sqrt{13}\ \text{m s}^{-1}$$

Its direction is given from Fig. 4.23 by

$$\tan\phi_1 = \frac{v_T}{v_R} = \frac{6}{4} = 1.5 \quad \text{where}$$

ϕ_1 is the angle which v makes with $\hat{\mathbf{r}}$.

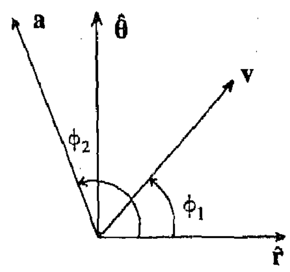


Fig.4.23

b) From Eq. 4.14 acceleration $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$

Since $\dot{r}$ and $\dot{\theta}$ are constant; $\ddot{r} = 0$, $\ddot{\theta} = \mathbf{0}$.

So $\mathbf{a} = (-r\dot{\theta}^2)\hat{\mathbf{r}} + 2\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}}$

$$= (-3\text{m} \times 4\ \text{rad}^2\text{s}^{-2})\hat{\mathbf{r}} + (2 \times 4\text{m s}^{-1} \times 2\ \text{rad}^2\text{s}^{-1})\hat{\boldsymbol{\theta}}$$

$$= -12\ \text{m s}^{-2}\,\hat{\mathbf{r}} + 16\ \text{m s}^{-2}\,\hat{\boldsymbol{\theta}}.$$

Its magnitude $a = \sqrt{(12 \times 12) + (16 \times 16)}\ \text{m s}^{-2} = 20\ \text{m s}^{-2}$ and its direction is given by $\tan\phi_2 = \dfrac{a_T}{a_R} = \left(-\dfrac{16}{12}\right) = -1.3$, where ϕ_2 is the angle which $\mathbf{a}$ makes with $\hat{\mathbf{r}}$ (see Fig. 4.23).

c) The kinetic energy of the particle is $K_{Rot} = \dfrac{1}{2}\mathbf{I}\,\omega^2$

$$= \frac{1}{2}mr^2\dot{\theta}^2$$

$$= \frac{1}{2}\left(\frac{5}{1000}\text{ kg}\right)\cdot(3\text{m})^2(2\text{ rad s}^{-1})^2$$

= 0.1 joule

3. Since $\mathbf{r} = 6t^4\hat{i} - 3t^2\hat{j} + (4t^3 - 5)\hat{k}$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 24t^3\hat{\mathbf{i}} - 6t\hat{\mathbf{j}} + 12t^2\hat{\mathbf{k}}$$

Angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v}$

or $\mathbf{L} = m[6t^4\hat{\mathbf{i}} - 3t^2\hat{\mathbf{j}} + (4t^3 - 5)\hat{\mathbf{k}}] \times [24t^3\hat{\mathbf{i}} - 6t\hat{\mathbf{j}} + 12t^2\hat{\mathbf{k}}]$

$= m[-36t^5\hat{\mathbf{k}} - 72t^6\hat{\mathbf{j}} + 72t^5\hat{\mathbf{k}} - 36t^4\hat{\mathbf{i}} + (96t^6 - 120t^3)\hat{\mathbf{j}} + (24t^4 - 30t)\hat{\mathbf{i}}]$

$= m[-(12t^4 + 30t)\hat{\mathbf{i}} + (24t^6 - 120t^3)\hat{\mathbf{j}} + 36t^5\hat{\mathbf{k}}]$

b) Torque $\tau = \frac{d\mathbf{L}}{dt} = m[-(48t^3 + 30)\hat{\mathbf{i}} + (144t^5 - 360t^2)\hat{\mathbf{j}} + 180t^4\hat{\mathbf{k}}]$

c) Kinetic energy of rotation $= \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}$

$= \frac{1}{2}m[576t^6 + 36t^2 + 144t^4]$

$= 18mt^2[16t^4 + 4t^2 + 1]$.

4. a) Refer to Fig. 4.24. The total angular momentum of the system

$= (0.02\text{ kg})(2\text{m s}^{-1})(0.5\text{ m}) + (0.03\text{ kg})(2\text{m s}^{-1})(0.5\text{m})$

$= 0.05\text{ kg m}^2\text{s}^{-1}$.

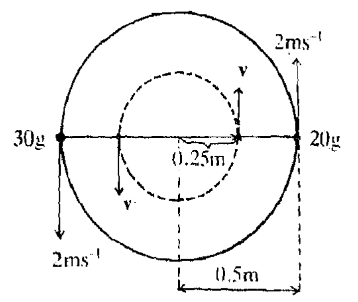


Fig. 4.24

b) Since no external torque acts on the system, the angular momentum remains conserved. As the rod is light, we shall assume it to be massless. As the particles remain connected by the rod the magnitudes of their velocities must be same (=v, say). When the rod gets contracted to half its original length the radius of the circular path (shown dotted) becomes 0.25 m (Fig. 4.24). So, the total angular momentum

$= (0.02\text{ kg})(v)(0.25\text{m}) + (0.03\text{ kg})(v)(0.25\text{m})$

$= 0.05 \times 0.25\, v\text{ kg m}$

From the principle of conservation of angular momentum, we get

$0.05\text{ kg m}^2\text{ s}^{-1} = 0.05 \times 0.25\ v\text{ kg m}$.

or $v = 4\text{m s}^{-1}$. So, speed of each particle becomes 4m s^{-1} i.e. double the original value.

UNIT 5 GRAVITATION

5.1 INTRODUCTION

In the previous four units you have studied linear as well as angular motion of a variety of objects. However, by and large we restricted our study to motion of objects on the earth. We did discuss some examples of motion of heavenly bodies but they lacked in details for want of the knowledge of gravitation. Therefore, we shall study gravitation in this unit.

We shall start from the familiar Kepler's laws of planetary motion to arrive at the law of universal gravitation. We shall then develop the concept of gravitational field and potential and use them to revisit the ideas of earth's gravity, and escape velocity. Finally, we shall visualise the gravitational force as a fundamental force in nature. Alongwith that we shall discuss, in brief, the electroweak and strong forces which are the other basic forces in nature.

In Block 2, we shall apply the concepts of mechanics developed in this block to motion under central conservative forces, systems of many particles and rigid bodies. We shall also study motion in accelerating frames of reference.

Objectives

After studying this unit you should be able to:

- apply the law of gravitation
- infer that the law of gravitation is universally true
- compute gravitational intensity and potential
- solve problems related to the variation of acceleration due to gravity with the height, depth and latitude of a place
- derive expression for velocity of escape
- distinguish between the fundamental forces in nature.

5.2 LAW OF GRAVITATION

You must be aware that the 'Law of Gravitation' was formulated by Sir Isaac Newton. The popular story goes like this:

Newton was sitting under a tree from which an apple fell and struck him on his head. This gave him the necessary impetus to discover the law. There could have been another part in the story: Newton was staring at the moon when the apple hit him (Fig. 5.1)! Newton's stroke of genius was that he realised that ***the force which causes apples to fall to the ground is of the same kind as the force which causes the moon to orbit the earth.*** In fact, the law of gravitation did not strike Newton in his first effort. He was looking for the answers to many questions related to wide-ranging topics from the 'Law of Falling Bodies' due to Galileo to Kepler's 'Laws of Planetary Motion'. Let us first arrive at the law of gravitation using Kepler's laws (Fig. 5.2). We shall then examine its universality

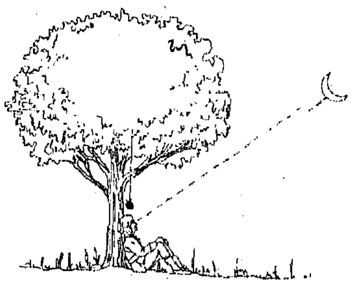


Fig. 5.1: Newton reelised that all objects in the universe whether on earth or in heavens move under the influence of the same force of gravity.